

Small Gröbner Fans of Ideals of Points

Elena Dimitrova¹

Department of Mathematical Sciences, Clemson University, Clemson, SC 29634, USA

Qijun He¹

Biocomplexity Institute, Virginia Tech, Blacksburg, VA 24061, USA

Lorenzo Robbiano

Dip. di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, I-16146 Genova, Italy

Brandilyn Stigler¹

Department of Mathematics, Southern Methodist University, Dallas, TX 75275, USA

Abstract

The main goal of this paper is to identify classes of ideals in affine K -algebras which have a unique reduced Gröbner basis. While problems arising from biology and statistics use mainly zero-dimensional radical ideals, we develop results for general ideals. We also provide a methodology for constructing such ideals. We then relax the condition of uniqueness and consider ideals with the same number of reduced Gröbner bases, that is, with the same cardinality of their associated Gröbner fan.

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1. Introduction

Gröbner bases have been used in a variety of applications. In one application, they are used to select minimal models for biological networks. Similar problems arise in the branch of statistics called Design of Experiments (see [10] and [7], Tutorial 92 for an introduction to this topic). In the context of a field K , functions which fit data in $\mathbb{X} \subseteq K^n$ lie in the coordinate ring $K[\mathbb{X}] := K[x_1, \dots, x_n]/\mathcal{I}(\mathbb{X})$.

Email addresses: edimit@clemson.edu (Elena Dimitrova), qijun@bi.vt.edu (Qijun He), robbiano@dima.unige.it (Lorenzo Robbiano), bstigler@smu.edu (Brandilyn Stigler)

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Then the coset $f + \mathcal{I}(\mathbb{X})$ describes the set of models which fit the input data in \mathbb{X} and one model is chosen by computing the normal form of $f \in K[x_1, \dots, x_n]$ with respect to a Gröbner basis of the ideal of points $\mathcal{I}(\mathbb{X})$. Changing term orderings results in potentially different normal forms, *i.e.*, different models, as is illustrated in the following example.

Example 1.1. Lactose metabolism in *E.coli* is controlled by the *lac* operon, a genetic system made up of simultaneously transcribed genes. It is said that the *lac* operon (x) is ON (lactose is metabolized) when the activating protein CAP (y) is present and when the inhibiting protein *lacI* (z) is absent. This behavior can be described by the Boolean function $f = y \wedge \neg z$; as a polynomial over the finite field \mathbb{F}_2 , we can write $f = y(z + 1) = yz + y$. If we consider the inputs $\mathbb{X} = \{(1, 0, 0), (0, 1, 0), (1, 0, 1)\}$ representing Boolean states for the *lac* operon, CAP, and *lacI* respectively, then the ideal of these points has two Gröbner bases, namely $\{x^2 + x, z^2 + z, y + x + 1, xz + z\}$ and $\{y^2 + y, z^2 + z, x + y + 1, yz\}$. The normal form of f is $yz + y$ and y respectively. Note that the function f is selected as a model using the first Gröbner basis while a simpler model is selected using the second Gröbner basis.

In [3] the authors raised the question of properties of \mathbb{X} that guarantee $\mathcal{I}(\mathbb{X})$ has a unique reduced Gröbner basis. The following example demonstrates that we must be careful in counting the number of reduced Gröbner bases.

Example 1.2. Let $I = \langle x + y + z \rangle \subset K[x, y, z]$, where K is any field. The reduced Gröbner bases of I are $\{x + y + z\}$, $\{x + z + y\}$, $\{y + x + z\}$, $\{y + z + x\}$, $\{z + x + y\}$, and $\{z + y + x\}$. Notice that the reduced Gröbner bases $\{x + y + z\}$ and $\{x + z + y\}$ are identified since the corresponding leading term ideals coincide. In fact, we say that I has three distinct reduced Gröbner bases, namely $\{x + y + z\}$, $\{y + x + z\}$, and $\{z + x + y\}$, corresponding to the three distinct leading term ideals.

Let us have a close look at the content of the paper. In Section 2 we introduce fundamental tools such as G-basic sets, GFan numbers, and linear shifts (see Definitions 2.2, 2.5, and 2.6). Then it is shown in Theorem 2.7 that ideals related by a linear shift share the same number of G-basic sets, equivalently the same GFan number. Finally the classical notion of an ideal of points is recalled together with the notion of a grid of points (see Definitions 2.8 and 2.9).

Section 3 starts with Theorem 3.3 which provides an easy characterisation of ideals whose GFan number is 1. Such ideals turn out to have also a unique basic set as shown in Corollary 3.5. Then the important notion of a distraction is recalled. It is shown that distractions and their linear shifts provide a large class of ideals with GFan number equal to 1 (see Theorem 3.10 and Corollary 3.14). The last subsection of this section focuses on natural distractions and associated staircases. Their strong connection is highlighted in Proposition 3.24.

Section 4 contains the most relevant results of the paper. Given a polynomial ring $P = K[x_1, \dots, x_n]$ we consider an ideal I in P generated by n univariate polynomials, one for each indeterminate. Definition 4.4 describes how two ideals

J_1 and J_2 which contain I can be considered to be complementary with respect to I , and the main Theorem 4.6 shows that complementary ideals have the same GFan number. Then Corollaries 4.9 and 4.11 provide good classes of complementary ideals.

The paper is concluded in Section 5 where some applications of the theory developed before and some hints to future research are illustrated.

Basic definitions and results are taken from [6], [7], and [8]. The entire paper is sprinkled with a generous number of examples. All of them were computed with CoCoA-5 (see [1]).

2. Background

In this section we let K be a field, let $P = K[x_1, \dots, x_n]$, and let I be an ideal in P . Then we introduce $\text{GFNum}(I)$ which is the number of reduced Gröbner bases of the ideal I , and start to investigate it.

We recall that a non-empty subset \mathcal{O} of \mathbb{T}^n is called an **order ideal** if it is closed under division (see [7], Definition 6.4.3). If σ is a term ordering, the set $\mathbb{T}^n \setminus \text{LT}_\sigma(I)$ is denoted by $\mathcal{O}_\sigma(I)$. It is well-known that $\mathcal{O}_\sigma(I)$ is an order ideal and the residue classes of its elements form a K -basis of P/I (see for instance [6], Corollary 2.4.11). It is also well-known that there are order ideals which are not of type $\mathcal{O}_\sigma(I)$; nevertheless the residue classes of their elements form a K -basis of P/I . The following example taken from [7] (see Example 6.4.2) is a case in point.

Example 2.1. Consider the ideal $I = \langle x^2 + xy + y^2, x^3, x^2y, xy^2, y^3 \rangle$ in $\mathbb{Q}[x, y]$. This ideal is symmetric with respect to switching x and y . Since the leading term of $x^2 + xy + y^2$ is either x^2 or y^2 , the ideal I has two possible leading term ideals, namely the ideals $J_1 = \langle x^2, xy^2, y^3 \rangle$ and $J_2 = \langle x^3, x^2y, y^2 \rangle$. Neither is symmetric. Thus they do not give rise to symmetric vector space bases of $\mathbb{Q}[x, y]/I$. However, the set of terms $\mathcal{O} = \{1, x, y, x^2, y^2\}$ is symmetric and represents a vector space basis of $\mathbb{Q}[x, y]/I$.

These considerations motivate the following definition.

Definition 2.2. An order ideal such that the classes of its elements form a K -basis of P/I is called a **basic set** for I . If it is of type $\mathcal{O}_\sigma(I)$ it is called a **G-basic set** for I .

Remark 2.3. Let I be a zero-dimensional ideal, let $s = \dim_K(P/I)$, let $\mathcal{O}_\sigma(I)$ be a G-basic set for I , and let \mathcal{O} be an order ideal with s elements. The normal forms of the elements in \mathcal{O} with respect to σ are linear combinations of the elements of $\mathcal{O}_\sigma(I)$, and hence can be represented by an $s \times s$ matrix. Clearly \mathcal{O} is a basic set for I if and only if such matrix is invertible.

For zero-dimensional ideals, basic sets are the main building blocks of the theory of border bases (see [7], Section 6.4 for the introduction to that theory) which is outside the scope of the present paper.

The notion of the Gröbner Fan of I was introduced in [9]. It is a subdivision of \mathbb{N}^n made with a finite number of polyhedral cones, such that the cones correspond one-to-one to the G-basic sets for I . As already observed in Remark 1.2, we must be careful in counting the number of reduced Gröbner bases of an ideal. So we are motivated to introduce the following definition.

Definition 2.4. Let I be an ideal in P . Two reduced Gröbner bases of I are said to be **G-basic equivalent** or simply **equivalent** if their associated G-basic sets are the same.

In the example examined in Remark 1.2, there are six reduced Gröbner bases but only three equivalence classes since there are only three G-basic sets associated to I . To specify the number of G-basic sets (equivalently the number of equivalence classes of reduced Gröbner bases), we introduce the following definition.

Definition 2.5. The number of G-basic sets for I is called the **GFan number** of I , and is denoted by $\text{GFNum}(I)$, since it coincides with the number of polyhedral cones in the Gröbner Fan of I .

Our next goal is to identify pairs of ideals which have the same GFan number. A first easy answer is obtained by considering linear shifts.

Definition 2.6. Let $a_1, \dots, a_n \in K \setminus \{0\}$, let $b_1, \dots, b_n \in K$, and let $\Phi : P \rightarrow P$ be the K -algebra homomorphism defined by $x_i \mapsto a_i x_i + b_i$ for $i = 1, \dots, n$. Then Φ is called a **linear shift** of P .

Theorem 2.7. Let Φ be a linear shift of P and let I be an ideal in P .

- (a) The ideals I and $\Phi(I)$ have the same G-basic sets.
- (b) We have $\text{GFNum}(I) = \text{GFNum}(\Phi(I))$.

Proof. Claim (b) clearly follows from (a), so let us prove (a). Let σ be a term ordering on \mathbb{T}^n and let f be a non-zero polynomial in I . It is clear that $\text{LT}_\sigma(f) = \text{LT}_\sigma(\Phi(f))$ which implies that $\text{LT}_\sigma(I) \subseteq \text{LT}_\sigma(\Phi(I))$. But Φ is an isomorphism and its inverse is also a linear shift, hence we get the other inclusion. Consequently we have $\text{LT}_\sigma(I) = \text{LT}_\sigma(\Phi(I))$ for every term ordering σ which implies that I and $\Phi(I)$ have the same G-basic sets, and the proof is complete. \square

Next we recall well-known facts about ideals of points. For an introduction to the topic see [7], Section 6.3.

Definition 2.8. A tuple $(c_1, \dots, c_n) \in K^n$ is called a **point**. It corresponds to the linear maximal ideal $\mathfrak{m} = \langle x_1 - c_1, \dots, x_n - c_n \rangle \in P$. The vanishing ideal $\mathcal{I}(\mathbb{Y})$ of a finite set \mathbb{Y} of points is of type $\mathcal{I}(\mathbb{Y}) = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_s$. It is a zero-dimensional radical ideal in P and we say that $\mathcal{I}(\mathbb{Y})$ is an **ideal of points**.

Definition 2.9. Let $d_1, \dots, d_n \in \mathbb{N}_+$ and for $i = 1, \dots, n$ let $(c_{i,1}, \dots, c_{i,d_i})$ be a d_i tuple of pairwise distinct elements of K . Then the set of points

$$\mathbb{X} = \{(c_{1,k_1}, \dots, c_{n,k_n}) \mid 1 \leq k_1 \leq d_1, \dots, 1 \leq k_n \leq d_n\}$$

is called a **grid of points** or a **full design** in K^n . It consists of $\prod_{i=1}^n d_i$ points.

Remark 2.10. Let \mathbb{Y} be a set of points. The vanishing ideal $\mathcal{I}(\mathbb{Y})$ of \mathbb{Y} is an ideal of points and for every $i = 1, \dots, n$ it contains monic univariate polynomials $g_i(x_i)$ which generate $\mathcal{I}(\mathbb{Y}) \cap K[x_i]$ (see [6], Proposition 3.7.1.c). These polynomials split as products of linear polynomials, *i.e.* we have $g_i(x_i) = \prod_{j=1}^{d_i} (x_i - c_{ij})$. Consequently the set of points \mathbb{Y} is contained in the grid \mathbb{X} whose vanishing ideal is $\mathcal{I}(\mathbb{X}) = \langle g_1(x_1), \dots, g_n(x_n) \rangle$.

Definition 2.11. The grid described above is called the **minimal grid** of \mathbb{Y} .

3. Ideals with One Reduced Gröbner Basis

In this section we look for conditions under which the GFan number of I is 1.

3.1. Algebraic Properties

To ease the reading, we introduce the following definition.

Definition 3.1. A polynomial $f \in P$ is called **factor-closed** if there exists $t \in \text{Supp}(f)$ such that all $t' \in \text{Supp}(f)$ have the property that t' divides t .

We are ready to prove the first result which produces an answer to our quest. First, we need an easy lemma.

Lemma 3.2. *Let I be an ideal in the polynomial ring P , let σ be a term ordering, let G be a minimal monic σ -Gröbner basis of I , and assume that every polynomial in G is factor-closed.*

(a) *The set G is the reduced σ -Gröbner basis of I .*

(b) *We have $\text{GFNum}(I) = 1$.*

Proof. Let us prove claim (a). For contradiction assume that G is not reduced. Since it is minimal and monic, there exist $i, j \in \{1, \dots, s\}$ and a power product $\tilde{t} \in \text{Supp}(g_i)$ such that $\text{LT}_\sigma(g_j) \mid \tilde{t}$. Since g_i is factor-closed we deduce that $\text{LT}_\sigma(g_j) \mid \text{LT}_\sigma(g_i)$, a contradiction to the minimality of G .

The proof of (b) follows from the observation that for every $i \in \{1, \dots, s\}$ the leading term of g_i is the same and every term ordering, hence G is the reduced Gröbner basis of I for every term ordering. \square

Theorem 3.3. *Let I be an ideal in the polynomial ring P . The following conditions are equivalent.*

- (a) *There exists a term ordering σ and a minimal monic σ -Gröbner basis G of I such that all the polynomials in G are factor-closed.*
- (b) *There exists a term ordering σ such that all the polynomials in the reduced σ -Gröbner basis of I are factor-closed.*
- (c) *We have $\text{GFNum}(I) = 1$.*

Proof. From the lemma we deduce that claims (a) and (b) are equivalent and that (a) \Rightarrow (c). Next we prove (c) \Rightarrow (b). By contradiction we assume that there exists i and a power product $\tilde{t} \in \text{Supp}(g_i)$ such that \tilde{t} does not divide $\text{LT}_\sigma(g_i)$. We let $t' = \tilde{t} / \gcd(\tilde{t}, \text{LT}_\sigma(g_i))$ and $t = \text{LT}_\sigma(g_i) / \gcd(\tilde{t}, \text{LT}_\sigma(g_i))$. Then t' and t are coprime and $t' \neq 1$. Therefore there exists x_j such that $x_j \mid t'$ and $x_j \nmid t$. Let τ be the lexicographic term ordering with $x_j >_\tau x_i$ for $i \neq j$. Then $\tilde{t} >_\tau \text{LT}_\sigma(g_i)$ and hence the reduced τ -Gröbner basis of I is different from G . This is a contradiction and the proof is complete. \square

Remark 3.4. In the recent preprint [4], related results are proved for so-called neural ideals, which are generated by certain factor-closed generalizations of monomials (pseudonomonials), in Boolean rings.

An interesting consequence of the theorem is described in the following corollary. Similar results are contained in [2], Section 2.

Corollary 3.5. *Let I be an ideal in P , assume that $\text{GFNum}(I) = 1$, and let $\mathcal{O}(I)$ be the unique G -basic set for I . Then the set $\mathcal{O}(I)$ is also the unique basic set for I .*

Proof. Let $G = \{g_1, \dots, g_s\}$ be the unique reduced Gröbner basis of I . For contradiction, assume that there exists a basic set \mathcal{O} for I such that $\mathcal{O} \neq \mathcal{O}(I)$, and let $t \in \mathcal{O} \setminus \mathcal{O}(I)$. By definition of Gröbner basis, there exists i such that $\text{LT}(g_i) \mid t$. From the theorem we know that g_i is factor-closed, hence every power product in $\text{Supp}(g_i)$ divides t . On the other hand \mathcal{O} is an order ideal, hence every power product in $\text{Supp}(g_i)$ is in \mathcal{O} . Therefore we get a non-trivial linear combination of elements of \mathcal{O} which is zero in P/I , thus a contradiction. \square

Corollary 3.6. *Let I be a monomial ideal in P , let $\mathcal{O}(I)$ be the set of power products which are not divisible by any power product in I , and let Φ be a linear shift of P .*

- (a) *We have $\text{GFNum}(I) = 1$ and $\mathcal{O}(I)$ is the unique basic set for I .*
- (b) *We have $\text{GFNum}(\Phi(I)) = 1$ and $\mathcal{O}(I)$ is the unique basic set for $\Phi(I)$.*

Proof. To prove claim (a) we observe that $\mathcal{O}(I)$ is the unique G -basic set for I by Theorem 3.3. Then the conclusion follows from Corollary 3.5.

Claim (b) follows from (a) and Theorem 2.7. \square

Remark 3.7. We observe that a linear shift is composed by two types of shifts, namely Φ_1 of type $x_i \mapsto a_i x_i$ and Φ_2 of type $x_i \mapsto x_i + b_i$. Clearly, if I is a monomial ideal we have $I = \Phi_1(I)$, so the only non-trivial part of Corollary 3.6 is that $\text{GFNum}(\Phi_2(I)) = 1$.

3.2. Distractions

In Corollary 3.6 we have seen a modification of monomial ideals which produces ideals with GFan number equal to 1. In the literature there is another interesting construction which yields the same result. Let us recall it.

We let K be a field. We choose n sequences π_1, \dots, π_n of elements of K in such a way that each sequence consists of **pairwise distinct elements**. Thus we let $\pi_i = (c_{i1}, c_{i2}, \dots)$ with $c_{ij} \in K$ and $c_{ij} \neq c_{ik}$ for $j \neq k$.

Definition 3.8. Let $\pi = (\pi_1, \dots, \pi_n)$.

- For every power product $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{T}^n$, the polynomial

$$D_\pi(t) = \prod_{i=1}^{\alpha_1} (x_1 - c_{1i}) \cdot \prod_{i=1}^{\alpha_2} (x_2 - c_{2i}) \cdots \prod_{i=1}^{\alpha_n} (x_n - c_{ni})$$

is called the **distraction** of t with respect to π .

- Let I be a monomial ideal in P , and let $\{t_1, \dots, t_s\}$ be the unique minimal monomial system of generators of I . Then we say that the ideal $D_\pi(I) = \langle D_\pi(t_1), \dots, D_\pi(t_s) \rangle$ is the **distraction** of I with respect to π .

Remark 3.9. In order to define the distraction $D_\pi(I)$, it suffices to specify the first $\max\{\deg_{x_i}(t_j) \mid j \in \{1, \dots, s\}\}$ elements of the sequence π_i . In particular, to distract a monomial ideal I , it is sufficient to use finite tuples of elements. Consequently we do not have to assume that K is infinite. It suffices that K has *sufficiently many* elements.

Let us recall a theorem from [7].

Theorem 3.10. *Let I be a monomial ideal in P , let $\{t_1, \dots, t_s\}$ be a minimal set of power products which generates I , let $\pi = (\pi_1, \dots, \pi_n)$ be sequences of pairwise distinct elements in K , and let $D_\pi(I) = \langle D_\pi(t_1), \dots, D_\pi(t_s) \rangle$ be the corresponding distraction of I .*

- (a) *The ideal $D_\pi(I)$ is radical.*
- (b) *The set $\{D_\pi(t_1), \dots, D_\pi(t_s)\}$ is the reduced σ -Gröbner basis of $D_\pi(I)$ for every term ordering σ .*
- (c) *We have $\text{GFNum}(D_\pi(I)) = 1$.*

Proof. See [7], Theorem 6.2.12. □

Let us look at some examples.

Example 3.11. Let $K = \mathbb{Q}$, $t_1 = x^3y$, $t_2 = x^2y^4$, and $I = \langle t_1, t_2 \rangle$. Consider $\pi_1 = (3, 2, 5)$, $\pi_2 = (2, -1, 3, 12)$. As the elements in π_1, π_2 are pairwise distinct, we can make the distraction of I with respect to $\pi = (\pi_1, \pi_2)$: $D_\pi(I) = \langle D_\pi(t_1), D_\pi(t_2) \rangle$ where

$$D_\pi(t_1) = (x - 3)(x - 2)(x - 5)(y - 2)$$

and

$$D_\pi(t_2) = (x-3)(x-2)(y-2)(y+1)(y-3)(y-12).$$

According to Theorem 3.10, the ideal $D_\pi(I)$ is radical, $\{D_\pi(t_1), D_\pi(t_2)\}$ is the reduced σ -Gröbner basis for every term ordering σ , and $\text{GFNum}(D_\pi(I)) = 1$.

While the previous example was computed over an infinite field, are there **finite fields** K where it is possible to distract the ideal I of Example 3.11?

Example 3.12. As the largest exponent is 4, we need a field which has at least four elements. So \mathbb{F}_2 and \mathbb{F}_3 are excluded. On the other hand, if $K = \mathbb{F}_5$, then we can choose $\pi = (\pi_1, \pi_2)$ where $\pi_1 = (1, 3, 0)$, $\pi_2 = (0, 1, 2, 3)$. Then we get $D_\pi(t_1) = (x-1)(x-3)xy$, $D_\pi(t_2) = (x-1)(x-3)y(y-1)(y-2)(y-3)$.

Example 3.13. As observed in Remark 2.10, given a set of points $\mathbb{Y} \in K^n$, its vanishing ideal $\mathcal{I}(\mathbb{Y})$ contains n univariate polynomials $f_i(x_i)$ which are products of linear polynomials of type $x_i - c_{ij}$, and define the minimal grid \mathbb{K} containing \mathbb{Y} . If $d_i = \deg(f_i(x_i))$, then $f_i(x_i)$ is the distraction of $x_i^{d_i}$ with respect to any permutation of the tuple of the c_{ij} .

From the above theorem we get the following result.

Corollary 3.14. *With the same assumptions as in the theorem, let I be a zero-dimensional ideal and let Φ be a linear shift of P .*

- (a) *The ideal $D_\pi(I)$ is an ideal of points and $\text{GFNum}(D_\pi(I)) = 1$.*
- (b) *The ideal $\Phi(D_\pi(I))$ is an ideal of points, and $\text{GFNum}(\Phi(D_\pi(I))) = 1$.*

Proof. Claim (a) follows from [7], Theorem 6.2.12.a and Theorem 3.10.c.

Claim (b) follows from (a) and Theorem 2.7. \square

Let us see an example which illustrates the corollary.

Example 3.15. If $K = \mathbb{Q}$, then $f_1 = x(x - \frac{1}{5})(x-2)(x+1)$ is the distraction of x^4 with respect to any permutation of the 4-tuple $(0, \frac{1}{5}, 2, -1)$. Likewise the polynomial $f_2 = y(y-1)(y-2)$ is the distraction of y^3 with respect to any permutation of the 3-tuple $(0, 1, 2)$. If we add x^2y and xy^2 to the monomial ideal generated by the set $\{x^4, y^3\}$, we get the monomial ideal $\mathfrak{a} = \langle x^4, y^3, x^2y, xy^2 \rangle$. A distraction of \mathfrak{a} is $I_1 = \langle f_1, f_2, f_3, f_4 \rangle$ where $f_3 = x(x - \frac{1}{5})y$, $f_4 = xy(y-1)$. In this case $\pi_1 = (0, \frac{1}{5}, 2, -1)$ and $\pi_2 = (0, 1, 2)$. According to the corollary, I_1 is an ideal of points. More precisely we have $I_1 = I(\mathbb{X})$ where $\mathbb{X} = \{(0, 0), (\frac{1}{5}, 0), (0, 1), (\frac{1}{5}, 1), (0, 2)\}$. See the following CoCoA-5 session.

```
Use P:=QQ[x,y];
A1:=x; A2:= x-2; A3:= x+1; A4:=x-(1/5);
B1:=y; B2:= y-1; B3:= y-2;
F1:= A1*A2*A3*A4; LT(F1);
F2:=B1*B2*B3; LT(F2);
F3:= A1*A4; LT(F3);
F4:= A1*B1*B2; LT(F4);
D1:=Ideal(F1,F2,F3,F4);
PD1:=PrimaryDecomposition0(D1); [ReducedGBasis(X) | X In PD1];
-- [[y, x], [y -1, x], [y -2, x], [y, x -1/5], [y -1, x -1/5]]
```


Another distraction of \mathfrak{a} is $I_2 = \langle f_1, f_2, g_3, g_4 \rangle$ where

$$\begin{aligned} f_3 &= (x-2)(x+1)(y-2) \\ f_4 &= (x-2)(y-2)(y-1) \end{aligned}$$

with $\pi_1 = (2, -1, 0, \frac{1}{5})$ and $\pi_2 = (2, 1, 0)$. A third distraction of \mathfrak{a} is the ideal $I_3 = \langle f_1, f_2, f_3, f_4 \rangle$ where

$$\begin{aligned} f_3 &= (x - \frac{1}{5})(x-2)(y-1) \\ f_4 &= (x - \frac{1}{5})(y-1)y \end{aligned}$$

with $\pi_1 = (\frac{1}{5}, 2, -1, 0)$ and $\pi_2 = (1, 0, 2)$.

However, if

$$\begin{aligned} f_3 &= (x-2)(y-1)(y-2) \\ f_4 &= (x+1)(x - \frac{1}{5})(y-1) \end{aligned}$$

then the ideal $I_4 = \langle f_1, f_2, f_3, f_4 \rangle$ is not a distraction of \mathfrak{a} , though it has the unique reduced Gröbner basis

$$\{x^4 - \frac{6}{5}x^3 - \frac{9}{5}x^2 + \frac{2}{5}x, y^2 - 3y + 2, x^2y - x^2 + \frac{4}{5}xy - \frac{4}{5}x - \frac{1}{5}y + \frac{1}{5}\}.$$

We observe that this is a distraction of $\mathfrak{b} = \langle x^4, y^2, x^2y \rangle$ since

$$\begin{aligned} x^4 - \frac{6}{5}x^3 - \frac{9}{5}x^2 + \frac{2}{5}x &= (x+1)(x - \frac{1}{5})(x)(x-2) \\ y^2 - 3y + 2 &= (y-1)(y-2) \\ x^2y - x^2 + \frac{4}{5}xy - \frac{4}{5}x - \frac{1}{5}y + \frac{1}{5} &= (y-1)(x+1)(x - \frac{1}{5}). \end{aligned}$$

But there are cases which are not distractions of any monomial ideal, and nevertheless have a unique reduced Gröbner basis and the “correct leading term ideal”, as the following example shows.

Example 3.16. Use `P:= QQ[x,y,z];`
`L:=mat([[0,0,0], [1,0,0], [1,1,0], [1,1,1]]);`
`I:=IdealOfPoints(P,L); I;`
`-- ideal(z^2 -z, y*z -z, x*z -z, y^2 -y, x*y -y, x^2 -x)`
`GF:=GroebnerFanIdeals(I);GF;Len(GF);`
`-- [ideal(z^2 -z, y*z -z, x*z -z, y^2 -y, x*y -y, x^2 -x)]`
`-- 1`

This is not a distraction of $\mathfrak{a} = \langle z^2, yz, xz, y^2, xy, x^2 \rangle$ since we have the equalities $yz - z = z(\mathbf{y} - \mathbf{1})$ and $xy = x\mathbf{y}$. However $\text{LT}_\sigma(I) = \mathfrak{a}$ for every σ .

A distraction of \mathfrak{a} is for instance $\langle z(z-1), yz, xz, y(y-1), xy, x(x-1) \rangle$. It is the ideal of the set of points $\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

3.3. Natural Distractions and Staircases

In this subsection we introduce an interesting family of distractions. First we recall some results from [7].

Definition 3.17. Let R be a ring. An ideal is called **irreducible** if it cannot be written as the intersection of two ideals, both of which properly contain it.

Proposition 3.18. Let K be a field and let $P = K[x_1, \dots, x_n]$.

- (a) Every proper ideal in P is a finite intersection of irreducible ideals.
- (b) A monomial ideal I in P is irreducible if and only if it is of the form $I = \langle x_{i_1}^{d_1}, \dots, x_{i_s}^{d_s} \rangle$ with $1 \leq i_1 < \dots < i_s \leq n$ and $d_1, \dots, d_s \in \mathbb{N}$.
- (c) A zero-dimensional monomial ideal is irreducible if and only if it is of the form $I = \langle x_1^{d_1}, \dots, x_n^{d_n} \rangle$ with $d_1, \dots, d_n \in \mathbb{N}$.

Proof. For claim (a) see [7], Proposition 5.6.17. For claim (b) see [7], Proposition 6.2.12. Claim (c) follows immediately from (b). \square

Example 3.19. Let $P = \mathbb{Q}[x, y, z]$ and let $I = \langle yz, y^2, x^2y, x^3, z^4, xz^3, x^2z^2 \rangle$. Then let $I_1 = \langle x, y, z^4 \rangle$, $I_2 = \langle x^2, y, z^3 \rangle$, $I_3 = \langle x^3, y, z^2 \rangle$, $I_4 = \langle x^2, y^2, z \rangle$. We have $I = I_1 \cap I_2 \cap I_3 \cap I_4$.

After Corollary 3.6 we know that associated to every monomial ideal there is a unique basic set $\mathcal{O}(I)$, the set of power products which are not divisible by any power product in I . This observation motivates the following definition.

Definition 3.20. Let K be a field and let $P = K[x_1, \dots, x_n]$.

1. Given $t = x_1^{a_1} \dots x_n^{a_n}$, we say that $p(t) = (a_1, \dots, a_n) \in K^n$ is the **point associated to t** .
2. Let I be a monomial ideal and let $\mathcal{O}(I)$ be the unique basic set associated to I . Then the set $\{p(t) \mid t \in \mathcal{O}(I)\}$ is called the **staircase** of points associated to I and denoted by $\text{Stair}(I)$.

Example 3.21. Let $I = \langle x^2, xyz^2, y^2, z^3 \rangle$. Then we have

$$\mathcal{O}(I) = \{1, z, z^2, y, yz, yz^2, x, xz, xz^2, xy, xyz\}$$

hence we get

$$\begin{aligned} \text{Stair}(I) = \{ & (0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 0), (0, 1, 1), (0, 1, 2), \\ & (1, 0, 0), (1, 0, 1), (1, 0, 2), (1, 1, 0), (1, 1, 1) \} \end{aligned}$$

Lemma 3.22. Let $P = K[x_1, \dots, x_n]$, let I_1, I_2 be zero-dimensional monomial ideals in P , let π_1, \dots, π_n be sequences of pairwise distinct elements of the field K , and let $\pi = (\pi_1, \dots, \pi_n)$.

- (a) $D_\pi(I_1 \cap I_2) = D_\pi(I_1) \cap D_\pi(I_2)$.

$$(b) \mathcal{O}(I_1 + I_2) = \mathcal{O}(I_1) \cap \mathcal{O}(I_2)$$

$$(c) \mathcal{O}(I_1 \cap I_2) = \mathcal{O}(I_1) \cup \mathcal{O}(I_2).$$

$$(d) \text{Stair}(I_1 \cap I_2) = \text{Stair}(I_1) \cup \text{Stair}(I_2).$$

Proof. Claim (a) follows from [7], Proposition 6.2.10. Let us prove claim (b). From the inclusions $I_1 \subseteq I_1 + I_2$ and $I_2 \subseteq I_1 + I_2$ we get $\mathcal{O}(I_1 + I_2) \subseteq \mathcal{O}(I_1)$ and $\mathcal{O}(I_1 + I_2) \subseteq \mathcal{O}(I_2)$, hence the inclusion $\mathcal{O}(I_1 + I_2) \subseteq \mathcal{O}(I_1) \cap \mathcal{O}(I_2)$. On the other hand, if t is a power product with $t \notin I_1$ and $t \notin I_2$, then $t \notin I_1 + I_2$ and the claim is proved.

Claim (d) follows immediately from (c) and the definition of staircase. Let us now prove (c). From the inclusions $I_1 \cap I_2 \subseteq I_1$ and $I_1 \cap I_2 \subseteq I_2$ we get the inclusions $\mathcal{O}(I_1) \subseteq \mathcal{O}(I_1 \cap I_2)$ and $\mathcal{O}(I_2) \subseteq \mathcal{O}(I_1 \cap I_2)$, hence the inclusion $\mathcal{O}(I_1) \cup \mathcal{O}(I_2) \subseteq \mathcal{O}(I_1 \cap I_2)$. To conclude the proof, we need to show that the two sets have the same number of elements. On the other hand, if I is a zero-dimensional monomial ideal, the number of elements of $\mathcal{O}(I)$ is finite. Since we have

$$\text{card}(\mathcal{O}(I_1) \cup \mathcal{O}(I_2)) = \text{card}(\mathcal{O}(I_1)) + \text{card}(\mathcal{O}(I_2)) - \text{card}(\mathcal{O}(I_1) \cap \mathcal{O}(I_2))$$

we need to prove the equality

$$\text{card}(\mathcal{O}(I_1 \cap I_2)) = \text{card}(\mathcal{O}(I_1)) + \text{card}(\mathcal{O}(I_2)) - \text{card}(\mathcal{O}(I_1) \cap \mathcal{O}(I_2)). \quad (*)$$

To show this equality we construct the exact sequence of K -vector spaces

$$0 \rightarrow P/(I_1 \cap I_2) \rightarrow (P/I_1) \oplus (P/I_2) \rightarrow P/(I_1 + I_2) \rightarrow 0$$

defined by the map

$$P/(I \cap J) \rightarrow (P/I) \oplus (P/J) \text{ given by } f + (I \cap J) \mapsto (f + I, f + J)$$

and the map

$$(P/I) \oplus (P/J) \rightarrow P/(I + J) \text{ given by } (f + I, g + J) \mapsto f - g + I + J.$$

From the exact sequence we get the equality

$$\text{card}(\mathcal{O}(I_1 \cap I_2)) = \text{card}(\mathcal{O}(I_1)) + \text{card}(\mathcal{O}(I_2)) - \text{card}(\mathcal{O}(I_1 + I_2))$$

and the conclusion follows from (b). \square

We are ready to introduce a special type of distractions.

Definition 3.23. Let I be a monomial ideal and let d_1, \dots, d_n be the maximal exponents of x_1, \dots, x_n in the minimal set of generators of I . Furthermore, let $\pi_{d_i} = (0, 1, 2, 3, \dots, d_i - 1)$ and $\pi_{\text{nat}} = (\pi_{d_1}, \dots, \pi_{d_n})$. Then the distraction $D_{\pi_{\text{nat}}}(I)$ is called the **natural distraction** (or classic distraction) of the ideal I .

Proposition 3.24. *Let K be a field, let $P = K[x_1, \dots, x_n]$, let I be a zero-dimensional monomial ideal, let d be the maximum exponent of the indeterminates in the minimal system of generators of I , and assume that $\text{char}(K) \geq d$. Then we have the equality $\mathcal{I}(\text{Stair}(I)) = D_{\pi_{\text{nat}}}(I)$.*

Proof. As a first step, we prove the claim with the extra assumption that I is irreducible, hence of type $I = \langle x_1^{a_1}, \dots, x_n^{a_n} \rangle$. In this case it is easy to see that $\text{Stair}(I) = \{(c_1, \dots, c_n) \mid 0 \leq c_k < a_k \text{ for } k = 1, \dots, n\}$. Consequently we have

$$\begin{aligned} \mathcal{I}(\text{Stair}(I)) &= \bigcap_{0 \leq c_k < a_k} \langle x_1 - c_1, \dots, x_n - c_n \rangle \\ &= \langle \prod_{c_1=1}^{a_1-1} (x_1 - c_1), \dots, \prod_{c_n=1}^{a_n-1} (x_n - c_n) \rangle \\ &= D_{\pi_{\text{nat}}}(I) \end{aligned}$$

Next we prove the general claim. From Proposition 3.18 we get an equality $I = \bigcap_{k=1}^s J_k$ with J_k irreducible for $k = 1, \dots, s$. We deduce the following equalities

$$\begin{aligned} \mathcal{I}(\text{Stair}(I)) &= \mathcal{I}(\text{Stair}(\bigcap_{k=1}^s J_k)) \stackrel{(1)}{=} \mathcal{I}(\bigcup_{k=1}^s \text{Stair}(J_k)) = \\ &= \bigcap_{k=1}^s \mathcal{I}(\text{Stair}(J_k)) \stackrel{(2)}{=} \bigcap_{k=1}^s D_{\pi_{\text{nat}}}(J_k) \stackrel{(3)}{=} D_{\pi_{\text{nat}}}(\bigcap_{k=1}^s J_k) = D_{\pi_{\text{nat}}}(I) \end{aligned}$$

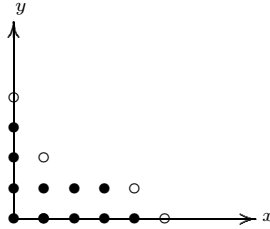
where equality (1) follows from Lemma 3.22.d, equality (2) follows from the special case discussed above, equality (3) follows from Lemma 3.22.a. \square

Let us see an example.

Example 3.25. Let $P = \mathbb{Q}[x, y]$ and let $I = \langle x^5, x^4y, xy^2, y^4 \rangle$. Then we have

$$D_{\pi_{\text{nat}}}(I) = \langle x(x-1)(x-2)(x-3)(x-4), x(x-1)(x-2)(x-3)y, \\ xy(y-1), y(y-1)(y-2)(y-3) \rangle$$

If we draw a picture of the power products involved, we get



where the white dots represent the generators of I and the black dots represents the power products in $\mathcal{O}(I)$. According to Proposition 3.24, we can check that the set of points defined by the ideal $D_{\pi_{\text{nat}}}(I)$ is exactly the staircase represented by the black dots.

4. Complementary ideals

In this section we concentrate on zero-dimensional ideals and introduce the notion of complementary ideals. We start by making an assumption which is kept for the entire section.

Assumptions 4.1. We let $d_1, \dots, d_n \in \mathbb{N}_+$, let K be a field, let P denote the polynomial ring $K[x_1, \dots, x_n]$, let $g_i(x_i) \in K[x_i]$ with $\deg(g_i) = d_i$ for $i = 1, \dots, n$, and let $I = \langle g_1, \dots, g_n \rangle$.

Lemma 4.2. With Assumptions 4.1 we have the following facts.

- (a) The set $\{g_1, \dots, g_n\}$ is the reduced σ -Gröbner basis of I for every term ordering σ , and hence $\text{GFNum}(I) = 1$.
- (b) The ideal I is zero-dimensional.
- (c) For every term ordering σ the residue class of $x_1^{d_1-1} x_2^{d_2-1} \dots x_n^{d_n-1}$ generates the socle of $P/\text{LT}_\sigma(I)$.
- (d) We have $\mathbb{T}^n \setminus \text{LT}_\sigma(I) = \{t \in \mathbb{T}^n \mid t \text{ divides } x_1^{d_1-1} x_2^{d_2-1} \dots x_n^{d_n-1}\}$ for every term ordering σ on \mathbb{T}^n .

Proof. Claim (a) follows from the fact that $\text{LT}_\sigma(g_i) = x_i^{d_i}$, hence they are pairwise coprime. Claim (b) follows from the Finiteness Criterion (see [6], Proposition 3.7.1). The other claims follow immediately. \square

Notation 4.3. This lemma suggests to denote $x_1^{d_1-1} x_2^{d_2-1} \dots x_n^{d_n-1}$ by t_{soc} . Moreover, if we let σ be a term ordering and let $H \supseteq I$ be an ideal in P , the set $\mathbb{T}^n \setminus \text{LT}_\sigma(H)$ is denoted by $B_\sigma(H)$. After Lemma 4.2.a we know that $B_\sigma(I)$ is the same for every σ , hence it will be denoted by $B(I)$.

Definition 4.4. With Assumptions 4.1 let $I = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_s$ be the primary decomposition of I , let $0 < t < s$, let $J_1 = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_t$, and let $J_2 = \mathfrak{q}_{t+1} \cap \dots \cap \mathfrak{q}_s$. Then we say that J_1, J_2 are **complementary ideals with respect to I** , or simply complementary ideals, if I is clear from the context.

The following lemma collects some elementary properties of complementary ideals.

Lemma 4.5. Let I satisfy Assumptions 4.1 and let J_1, J_2 be complementary ideals with respect to I .

- (a) $I = J_1 \cap J_2$, $J_1 + J_2 = \langle 1 \rangle$, $J_2 = I : J_1$, and $J_1 = I : J_2$.
- (b) There is an isomorphism of K -algebras $\varphi : P/I \cong P/J_1 \times P/J_2$.
- (c) $\dim_K(P/I) = \dim_K(P/J_1) + \dim_K(P/J_2)$, and hence $\text{card}(B(I)) = \text{card}(B_\sigma(J_1)) + \text{card}(B_\sigma(J_2))$.

Proof. Claim (a) is easy to prove using standard facts in commutative algebra. Claim (b) follows from (a) and the Chinese Remainder Theorem (see for instance [6], Lemma 3.7.4). Claim (c) follows from (b) since the residue classes of the elements of $B_\sigma(H)$ form a K -basis of P/H for any term ordering σ and any ideal H in P . \square

Theorem 4.6. *Let $d_1, \dots, d_n \in \mathbb{N}_+$, let K be a field, let $P = K[x_1, \dots, x_n]$, let $g_i(x_i) \in K[x_i]$, let $I = \langle g_1, \dots, g_n \rangle$, and finally let J_1, J_2 be complementary ideals with respect to I .*

- (a) *For $i = 1, 2$ we have $B(I) \cap \text{LT}_\sigma(J_i) = B(I) \setminus B_\sigma(J_i)$.*
- (b) *Let $\alpha : B(I) \cap \text{LT}_\sigma(J_1) \rightarrow \mathbb{T}^n$ be the map which sends t to t_{soc}/t . Then α is injective and induces a map $\vartheta : B(I) \setminus B_\sigma(J_1) \rightarrow B_\sigma(J_2)$ which is bijective.*
- (c) *There is a bijection between the set of reduced Gröbner bases of J_1 and the set of reduced Gröbner bases of J_2 .*
- (d) *We have $\text{GFNum}(I) = \text{GFNum}(J)$.*

Proof. First, we prove claim (a). From $J_i \supseteq I$ we deduce that $\text{LT}(J_i) \supseteq \text{LT}(I)$, hence $B_\sigma(J_i) \subseteq B(I)$ which implies the claim.

Next we prove claim (b). The fact that α is injective is obvious. By contradiction assume that $t_{\text{soc}}/t \notin B_\sigma(J_2)$. Then $t_{\text{soc}}/t \in \text{LT}_\sigma(J_2)$. We have $t_{\text{soc}} = t \cdot t_{\text{soc}}/t \in \text{LT}_\sigma(J_1) \cdot \text{LT}_\sigma(J_2) \subseteq \text{LT}_\sigma(J_1 \cdot J_2)$. Clearly $J_1 \cdot J_2 \subseteq I$, and so we get $t_{\text{soc}} \in \text{LT}_\sigma(I)$ which yields a contradiction by Lemma 4.2.a. Consequently we get an injective map $B(I) \cap \text{LT}_\sigma(J_1) \rightarrow B_\sigma(J_2)$, and from (a) we can rewrite it as a map $\vartheta : B(I) \setminus B_\sigma(J_1) \rightarrow B_\sigma(J_2)$ which is injective. Lemma 4.5.(c) shows that the two sets have the same cardinality, hence we conclude that ϑ is bijective.

Since (d) follows immediately from (c), let us prove claim (c). For $i = 1, 2$ we let \mathcal{G}_i be the set of reduced Gröbner bases of J_i . We define a map $\alpha : \mathcal{G}_1 \rightarrow \mathcal{G}_2$ as follows. Let σ be a term ordering and let G_σ be the reduced σ -Gröbner basis of J_1 . It identifies the set $B_\sigma(J_1)$ which, in turn, identifies the set $B_\sigma(J_2)$ via the map ϑ . The set $B_\sigma(J_2)$ identifies $\text{LT}_\sigma(J_2)$ which, in turn, identifies the reduced σ -Gröbner basis of J_2 by the uniqueness of the reduced σ -Gröbner basis of J_2 . So α is defined. Interchanging the roles of J_1 and J_2 we define a map $\beta : \mathcal{G}_2 \rightarrow \mathcal{G}_1$ in the same way, and it is clear from the construction that α and β are inverses to each other. \square

Remark 4.7. In the proof of claim (b) we used the fact that $J_1 \cdot J_2 \subseteq I$. Actually we have $J_1 \cdot J_2 = I$ (see for instance [8], Theorem 2.2.1).

Let us illustrate the theorem with an example.

Example 4.8. `K:=QQ; Use P:=K[x,y];
I:=ideal(x*(x^2+1)^2*(x-1), (y^3-1)*(y+2));
PD:=PrimaryDecomposition0(I);[ReducedGBasis(X) | X In PD];
/*
[[y +2, x], [y -1, x], [x, y^2 +y +1], [y +2, x -1],`

```

[y -1, x -1], [x -1, y^2 +y +1], [y +2, x^4 +2*x^2 +1],
[y -1, x^4 +2*x^2 +1], [y^2 +y +1, x^4 +2*x^2 +1]]
*/
J1:= Intersection(ideal(x -1, y^2 +y +1), ideal(y +2, x),
ideal(y +2, x^4 +2*x^2 +1));
ReducedGBasis(J1); QB1:=QuotientBasisSorted(J1); QB1;
-- [x*y +2*x -y -2, y^3 +3*y^2 +3*y +2,
--      x^5 +2*x^3 +(4/3)*y^2 +x +(4/3)*y -8/3]
-- [1, y, x, y^2, x^2, x^3, x^4]
J2:=Intersection(ideal(x, y^2 +y +1), ideal(y -1, x -1),
ideal(y -1, x), ideal(y +2, x -1), ideal(y -1, x^4 +2*x^2 +1),
ideal(y^2 +y +1, x^4 +2*x^2 +1));
indent(ReducedGBasis(J2)); QB2:=QuotientBasisSorted(J2); QB2;
/*[
y^4 +2*y^3 -y -2,
x*y^3 -y^3 -x +1,
x^5*y -x^5 +2*x^3*y -2*x^3 +(-4/3)*y^3 +x*y -x +4/3,
x^6 -x^5 +2*x^4 -2*x^3 +x^2 -x
] */
-- [1, y, x, y^2, x*y, x^2, y^3, x*y^2, x^2*y, x^3, x^2*y^2,
--      x^3*y, x^4, x^3*y^2, x^4*y, x^5, x^4*y^2]
multiplicity(P/I); multiplicity(P/J1); multiplicity(P/J2);
-- 24
-- 7
-- 17
GF1:=GroebnerFanIdeals(J1);indent(GF1);
/*
ideal(x*y +2*x -y -2, y^3 +3*y^2 +3*y +2,
x^5 +2*x^3 +(4/3)*y^2 +x +(4/3)*y -8/3),
ideal(x*y -y +2*x -2, y^2 +(3/4)*x^5 +(3/2)*x^3 +y +(3/4)*x -2,
x^6 -x^5 +2*x^4 -2*x^3 +x^2 -x)
*/
GF2:=GroebnerFanIdeals(J2);indent(GF2);
/*
ideal(y^4 +2*y^3 -y -2, x*y^3 -y^3 -x +1,
x^5*y -x^5 +2*x^3*y -2*x^3 +(-4/3)*y^3 +x*y -x +4/3,
x^6 -x^5 +2*x^4 -2*x^3 +x^2 -x),
ideal(y^3 +(-3/4)*x^5*y +(-3/2)*x^3*y +(3/4)*x^5 +(-3/4)*x*y +(3/2)*x^3 +(3/4)*x -1,
x^5*y^2 +2*x^3*y^2 +x^5*y +x*y^2 +2*x^3*y -2*x^5 +x*y -4*x^3 -2*x,
x^6 -x^5 +2*x^4 -2*x^3 +x^2 -x)
*/

```

Next we detect interesting situations where the theorem applies.

Corollary 4.9. *With Assumptions 4.1, let I be radical and let J be an ideal in P such that $J \supseteq I$.*

- (a) *The ideals J and $I : J$ are radical.*
- (b) *The ideals J and $I : J$ are complementary ideals with respect to I , and hence the conclusions of Theorem 4.6 apply to $J_1 = J$ and $J_2 = I : J$.*

Proof. Let us prove claim (a). As the ideal I is zero-dimensional and radical, there are maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ in P such that $I = \bigcap_{i=1}^s \mathfrak{m}_i$. Let $S = \{1, \dots, s\}$. The Chinese Remainder Theorem implies that there is an isomorphism $P/I \cong \prod_{i \in S} P/\mathfrak{m}_i$. Via this isomorphism the ideal J is sent to s ideals which are either $\langle 0 \rangle$ or $\langle 1 \rangle$. Let T be the subset of S of indices which correspond to the zero ideals. Then $J = \bigcap_{i \in T} \mathfrak{m}_i$ and hence it is radical. The same proof works for $I : J$.

To prove claim (b), it suffices to observe that we have the equality $I : J = \bigcap_{i \in S \setminus T} \mathfrak{m}_i$. \square

The following example illustrates this corollary.

Example 4.10. `K:=QQ; Use P:=K[x,y];
I:=ideal((x^2+1)*(x-1)*(x-2), (y^2-2)*(y+2));
J1:= I+ideal(x-1+y^2-2);
J2:=Colon(I,J1);
ReducedGBasis(J1); QB1:=QuotientBasis(J1); QB1;
-- [x -1, y^2 -2]
-- [1, y]
ReducedGBasis(J2); QB2:=QuotientBasis(J2); QB2;
-- [y^3 +2*y^2 -2*y -4,
x^3*y +2*x^3 -2*x^2*y -4*x^2 +x*y +2*x -2*y -4,
x^4 -3*x^3 +3*x^2 -3*x +2]
-- [1, y, y^2, x, x*y, x*y^2, x^2, x^2*y, x^2*y^2, x^3]
multiplicity(P/I); multiplicity(P/J1); multiplicity(P/J2);
-- 12
-- 2
-- 10
GF1:=GroebnerFanIdeals(J1);
GF2:=GroebnerFanIdeals(J2);
Len(GF1);Len(GF2);
-- 1
-- 1
indent([ReducedGBasis(I) | I in GF1]);
-- [x -1, y^2 -2]
indent([ReducedGBasis(I) | I in GF2]);
-- [y^3 +2*y^2 -2*y -4,
x^3*y +2*x^3 -2*x^2*y -4*x^2 +x*y +2*x -2*y -4,
x^4 -3*x^3 +3*x^2 -3*x +2]`

Returning to grids of points, we arrive at the following result.

Corollary 4.11. *For every grid \mathbb{X} , Corollary 4.9 applies to $\mathcal{I} = \mathcal{I}(\mathbb{X})$.*

Proof. Let \mathbb{X} be a grid of points in K^n as defined above and for $i = 1, \dots, n$, let $g_i = \prod_{j=1}^{d_i} (x_i - c_{ij})$. Then the vanishing ideal of \mathbb{X} is $\mathcal{I}(\mathbb{X}) = \langle g_1, \dots, g_n \rangle$. The ideal $\mathcal{I}(\mathbb{X})$ is clearly radical, and every ideal which contains $\mathcal{I}(\mathbb{X})$ is the vanishing ideal of a subset \mathbb{Y} of \mathbb{X} , i.e. it is of type $\mathcal{I}(\mathbb{Y})$. Consequently we have $\mathcal{I}(\mathbb{X}) : \mathcal{I}(\mathbb{Y}) = \mathcal{I}(\mathbb{X} \setminus \mathbb{Y})$. \square

Remark 4.12. We observe that every set of points \mathbb{Y} can be embedded into a unique minimal grid \mathbb{X} , hence it produces an ideal $\mathcal{I}(\mathbb{X})$, an ideal $\mathcal{I}(\mathbb{Y})$, and the ideal $\mathcal{I}(\mathbb{X}) : \mathcal{I}(\mathbb{Y})$ as above.

Remark 4.13. Let K be a finite field and let p be its characteristic. It is known that K is a finite-dimensional \mathbb{F}_p -vector space, hence the number of its elements is $q = p^e$ where $e = \dim_{\mathbb{F}_p}(K)$. Given an indeterminate z the univariate polynomial $z^q - z$ is called a **field equation** of K since it is known that $x^q - z = \prod_{a \in K} (x - a)$ (see [5], Section 4.13).

Consequently, if $P = K[x_1, \dots, x_n]$ and $g_i = x_i^q - x_i$ for $i = 1, \dots, n$, then the ideal $\langle g_1, \dots, g_n \rangle$ is the vanishing ideal of a grid and the above Corollary 4.9 applies to this case, as observed in Corollary 4.11.

Let us see an example with $K = \mathbb{F}_3$.

Example 4.14. `K:=ZZ/(3); Use P:= K[x,y,z];
I:=ideal(x^3-x, y^3-y, z^3-z);`


```

J1:= I+ideal(x^2-y-z);
J2:=Colon(I, J1);

ReducedGBasis(J1); QB1:=QuotientBasis(J1); QB1;
-- [y^2 -y*z +z^2 -y -z, x*y +x*z -x, x^2 -y -z, z^3 -z]
-- [1, z, z^2, y, y*z, y*z^2, x, x*z, x*z^2]

ReducedGBasis(J2); QB2:=QuotientBasis(J2); QB2;
-- [z^3 -z, y^3 -y, x*y^2 -x*y*z +x*z^2 +x*y +x*z,
--   x^2*y +x^2*z +x^2 +y^2 -y*z +z^2 -1, x^3 -x]
-- [1, z, z^2, y, y*z, y*z^2, y^2, y^2*z, y^2*z^2, x, x*z,
--   x*z^2, x*y, x*y*z, x*y*z^2, x^2, x^2*z, x^2*z^2]
multiplicity(P/I); multiplicity(P/J1); multiplicity(P/J2);
-- 27
-- 9
-- 18
GF1:=GroebnerFanIdeals(J1);
GF2:=GroebnerFanIdeals(J2);
Len(GF1);Len(GF2);
-- 4
-- 4
indent([ReducedGBasis(I) | I in GF1]);
/*
[y^2 -y*z +z^2 -y -z, x*y +x*z -x, x^2 -y -z, z^3 -z],
[x^2 -z -y, x*z +x*y -x, z^2 -y*z +y^2 -z -y, y^3 -y],
[y -x^2 +z, x^3 -x, z^3 -z],
[x^3 -x, z +y -x^2, y^3 -y]
*/
indent([ReducedGBasis(I) | I in GF2]);
/*
[z^3 -z, y^3 -y, x*y^2 -x*y*z +x*z^2 +x*y +x*z,
x^2*y +x^2*z +x^2 +y^2 -y*z +z^2 -1, x^3 -x],
[y^3 -y, x^3 -x, x^2*z +x^2*y +z^2 +x^2 -y*z +y^2 -1,
x*z^2 -x*y*z +x*y^2 +x*z +x*y, z^3 -z],
[x^3 -x, z^3 -z, y^2 +x^2*y -y*z +x^2*z +z^2 +x^2 -1],
[x^3 -x, z^2 -y*z +y^2 +x^2*z +x^2*y +x^2 -1, y^3 -y]
*/

```

5. Applications, Remarks, and a Problem

Theorem 4.6 shows, among other results, that complementary ideals have the same number of reduced Gröbner bases. The advantage of this is that it may be computationally easy to test whether a *small* set of data has a unique Gröbner basis associated to it and then to generate a *larger* set via the complement. Let us see an easy application of this remark.

Proposition 5.1. *Let \mathbb{X}, \mathbb{Y} be grids of points such that $\mathbb{Y} \subset \mathbb{X}$, and let J be the vanishing ideal of $\mathbb{X} \setminus \mathbb{Y}$.*

(a) *We have $\text{GFNum}(J) = 1$.*

(b) *In particular, statement (a) holds for a single point \mathbb{Y} .*

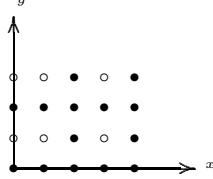
Proof. As (b) is a special case of (a), let us prove claim (a). Since \mathbb{Y} is a grid of points, we get $\text{GFNum}(\mathcal{I}(\mathbb{Y})) = 1$ from Lemma 4.2.a, and the conclusion follows from Theorem 4.6.d. \square

Example 5.2. Use $P := \mathbb{Q}\mathbb{Q}[x,y]$;

```

F:=x*(x-1)*(x-2)*(x-3)*(x-4);
G:=y*(y-1)*(y-2)*(y-3);
I:=ideal(F,G);
-----
M:=mat([[0,1], [0,3], [1,1], [1,3], [3,1], [3,3]]);
J1:=IdealOfPoints(P,M);
J2:=Colon(I,J1);
GF:=GroebnerFanIdeals(J2);GF;
-- [ideal(x^2*y^2 -2*x^2*y -6*x*y^2 +12*x*y +8*y^2 -16*y,
--      y^4 -6*y^3 +11*y^2 -6*y, x^5 -10*x^4 +35*x^3 -50*x^2 +24*x)]
Len(GF);
-- 1
-- The ideal J1 is the vanishing ideal of the "white dots".
-- The ideal J2 is the vanishing ideal of the "black dots".

```



The following remark indicates a possible direction of future research. In particular it gives a hint on how to generalize the notion of a distraction.

Remark 5.3. In Definition 3.8 we can substitute the sequences of elements in K with sequences of linear polynomials of type $x_i - c_{ij}$. But then we can also drop the assumption that these polynomials are linear. This generalization for instance would allow us to compute distractions even in small fields where there are possibly not enough linear polynomials.

Example 5.4. $K := \mathbb{Z}\mathbb{Z}/(2)$;

```

Use P:= K[x,y];
A:= ideal(x^5, x^2*y^4, y^7);
F1:= (x^2+x+1)*(x^3+x+1);
F2:= (y^3+y^2+1)*(y^4+y+1);
F3:= (x^2+x+1)*(y^4+y+1);
I:=ideal(F1,F2,F3);
LT(I) = A;
-- true
GF:=GroebnerFanIdeals(I);
RGF := [ReducedGBasis(I) | I in GF]; RGF; Len(RGF);
-- [[x^5 +x^4 +1, x^2*y^4 +x*y^4 +y^4 +x^2*y +x^2 +x*y +x +y +1,
--      y^7 +y^6 +y^2 +y +1]]
-- 1

```

The ideal I is a generalized distraction of $\langle x^5, x^2y^4, y^7 \rangle$ in the sense described in Remark 5.3.

We end the paper with a problem which is still open. Although distractions and their linear shifts provide a huge amount of ideals of points with GFan number 1, Example 3.16 shows that they do not cover all ideals of points with this property. So one problem is still open.

Problem. Characterize geometrically the sets of points with a unique reduced Gröbner basis.

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